Pergamon

Computers Math. Applic. Vol. 30, No. 3–6, pp. 83–101, 1995

0898-1221/95 \$9.50 + 0.00

0898-1221(95)00088-7

Local Smoothness of Functions and Bernstein-Durrmeyer Operators

H. H. GONSKA[†] AND D.-X. ZHOU[‡]
Department of Mathematics
University of Duisburg
D-47057 Duisburg, Germany
gonska@marvin.uni-duisburg.de
Dzhou@marvin.uni-duisburg.de

Abstract—Continuous functions satisfying a local *Lipschitz* condition of order $\alpha (0 < \alpha < 1)$ on any subset of $[0,1]$ are characterized by the local rate of convergence of Bernstein-Durrmeyer operators. As an application, we give an algorithm for singular detection. A new direct estimate for the approximation of continuous functions by Bernstein-Durrmeyer operators and Kantorovich operators is also presented.

Keywords—Local Lipschitz conditions, Bernstein-Durrmeyer operators, Singular detection, Kantorovich operators, Bernstein type operators.

1. INTRODUCTION AND MAIN RESULTS

The Bernstein-Durrmeyer operators on $[0,1]$ are defined as

$$D_n(f,x) = \sum_{k=0}^{n} P_{n,k}(x)(n+1)\int_0^1 f(t)P_{n,k}(t)\,dt, \qquad x \in [0,1], \tag{1.1}$$

where $f \in L_1[0,1]$ and, for $0 \le k \le n, P_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$.

These operators are very interesting approximation processes and have many nice properties such as commutativity. Their approximation rates are closely related to smoothness properties of the function they approximate. This fact has been shown in many recent papers; see [1–8]. The final characterization of the convergence in $L_p (1 \le p \le \infty)$ was given in terms of K-functionals and the so-called Ditzian-Totik moduli of smoothness as follows:

$$C^{-1}K\left(f, n^{-1/2}\right)_p \le \|D_n(f) - f\|_p \le CK\left(f, n^{-1/2}\right)_p, \tag{1.2}$$

where C is a positive constant depending only on $1 \le p \le \infty$ and $K(f,t)_p$ is the K-functional given by

$$\begin{aligned} K(f,t)_p &= \inf_{g \in C^2[0,1]} \left\{ \|f-g\|_p + t^2 \left\|\left(\varphi^2 g'\right)'\right\|_p \right\}, \\ \varphi(x) &= \sqrt{x(1-x)}. \end{aligned} \tag{1.3}$$

[†]Supported in part under NATO Grant CRG. 891013.
[‡]The second author is supported by the Alexander-von-Humboldt Foundation. He is on leave from: Institute of Mathematics, Academia Sinica, Beijing, 100080, People's Republic of China.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

On the other side, we can use the Bernstein-Durrmeyer operators to characterize the Lip α functions on $[0,1]$ with $0<\alpha<1$. In fact, it was proved in [7] that for $f\in C[0,1]$, $0<\alpha<1$,

$$|D_n(f,x)-f(x)|\le M_f\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right)^{\alpha/2},\qquad n\in\mathbb{N},\quad x\in[0,1] \tag{1.4}$$

with a constant M_f independent of n and x, if and only if $\omega_1(f,t)=O(t^\alpha)$, where

$$\begin{aligned}&\omega_1(f,t)=\sup_{0<h\le t}\|\Delta_h f(x)\|_\infty,\\ &\Delta_h f(x)=f(x+h)-f(x),\ \text{ if } x,x+h\in[0,1];\\ &\Delta_h f(x)=0,\ \text{otherwise.}\end{aligned} \tag{1.5}$$

All the above-mentioned results deal with global smoothness of functions and global approximation properties. The purpose of this paper is to give an equivalence between local smoothness of functions and local convergence of Bernstein-Durrmeyer operators as follows.

THEOREM 1. *Let $f\in C[0,1]$, $D_n(f,x)$ be given by (1.1), $0<\alpha<1$, and E be any subset of $[0,1]$. Then we have*

$$|f(x)-f(y)|\le M_f|x-y|^\alpha,\qquad x\in[0,1],\quad y\in E, \tag{1.6}$$

if and only if

$$|D_n(f,x)-f(x)|\le M'_f\left(\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right)^{\alpha/2}+(d(x,E))^\alpha\right),\qquad \begin{array}{l}x\in[0,1],\\ n\in\mathbb{N},\end{array} \tag{1.7}$$

where M_f and M'_f are constants depending only on α and f, and where $d(x,E)$ is the distance between x and E defined by

$$d(x,E)=\inf_{y\in E}\{|x-y|\}. \tag{1.8}$$

We say that a continuous function f is locally Lip $\alpha(0<\alpha\le1)$ on E if it satisfies condition (1.6). In particular, when $E=\{x_0\}\subset[0,1]$ and $f\in C[0,1]$ satisfies (1.6) with $0<\alpha\le1$, we say that f is locally Lip α at the point x_0. In this case, we can easily obtain the following corollary from Theorem 1.

COROLLARY. *Let $f\in C[0,1]$, $0<\alpha<1$, $x_0\in[0,1]$. Then f is locally* Lip α *at the point x_0 if and only if*

(i)
$$|D_n(f,x)-f(x)|\le M_f\left(n^{-\alpha/2}+|x-x_0|^\alpha\right),\qquad x\in[0,1],\quad n\in\mathbb{N}, \tag{1.9}$$

when $x_0\in(0,1)$.

(ii)
$$|D_n(f,x)-f(x)|\le M_f\left(n^{-\alpha}+|x-x_0|^\alpha\right),\qquad x\in[0,1],\quad n\in\mathbb{N}, \tag{1.10}$$

when $x_0=0$ or $=1$.

This corollary has an interesting application in signal processing. It can be used for singular detection. We say that a continuous signal $f\in C[0,1]$ has a singular point x_0 with Hölder exponent $\alpha(0<\alpha<1)$ if f is locally Lip α at x_0 but is not locally Lip β at x_0 for any $\beta>\alpha$. Thus, we can give an algorithm for singular detection by means of Bernstein-Durrmeyer operators, which is the main content of Section 4. Let us mention that some estimates of the local approximation rate by means of local maximal functions have been given by Lenze [9].

Note that Theorem 1 states the equivalence only in the case $0<\alpha<1$. It is then natural to investigate the case $\alpha=1$. However, things are quite different in this case, as shown in the following theorem.

THEOREM 2. *For any nonempty subset E of $[0,1]$ whose closure does not contain any endpoint, there exists a function $f \in C[0,1]$ which satisfies* (1.7) *but does not satisfy* (1.6) *for $\alpha = 1$.*

To prove the above theorem, we need a direct estimate for the Bernstein-Durrmeyer operators in $C[0,1]$ as follows.

THEOREM 3. *Let $f \in C[0,1], n \in \mathbb{N}$. Then*

$$\|D_n(f) - f\|_\infty \le \frac{M}{n}\left\{\int_{1/\sqrt{n}}^{1/2} \frac{\omega_\varphi^2(f,t)_\infty}{t^3}\,dt + E_0(f)_\infty\right\}, \tag{1.11}$$

where M is a positive constant independent of n and f, $E_0(f)_\infty$ is the distance between f and the space of constant functions on $[0,1]$ under the L_∞-norm, $\omega_\varphi^2(f,t)_\infty$ is the Ditzian-Totik modulus of smoothness defined as

$$\begin{aligned} \omega_\varphi^2(f,t)_\infty &= \sup\left\{\left|\Delta_{h\varphi(x)}^2 f(x)\right| : 0 < h \le t,\, x \in [0,1]\right\}, \\ \Delta_h^2 f(x) &= f(x+h) - 2f(x) + f(x-h), \\ &\qquad \text{whenever } x \pm h \in [0,1]; \\ \Delta_h^2 f(x) &= 0, \ \text{otherwise}. \end{aligned} \tag{1.12}$$

Theorem 3 is also of some value in the investigation of global approximation. The approximation of Bernstein-Durrmeyer operators in the space $L_p[0,1]$ with $1 < p < \infty$ has been completely characterized by the Ditzian-Totik modulus of smoothness (see [3,6]). In the space $C[0,1]$ of continuous functions, things are somewhat different. In [7] the second author has given the direct estimate

$$\|D_n(f) - f\|_\infty \le M\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty + \omega_1\left(f, \frac{1}{n}\right) + \frac{\|f\|_\infty}{n}\right). \tag{1.13}$$

But this estimate involves both the classical modulus of continuity and the Ditzian-Totik modulus of smoothness. Theorem 3 is somewhat better for our purpose because it uses only the latter one.

By the same method as for Theorem 3, we can give a similar direct estimate for Kantorovich operators. The Kantorovich operators on $L_1[0,1]$ are defined by

$$\begin{gathered} K_n(f,x) = \sum_{k=0}^{n} P_{n,k}(x)(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt, \\ x \in [0,1], \quad f \in L_1[0,1]. \end{gathered} \tag{1.14}$$

For these operators, we can state a similar direct estimate as follows. The proof of this estimate is almost the same as (1.11) and we omit it.

THEOREM 4. *Let $f \in C[0,1]$, $n \in \mathbb{N}$. Then*

$$\|K_n(f) - f\|_\infty \le \frac{M}{n}\left\{\int_{1/\sqrt{n}}^{1/2} \frac{\omega_\varphi^2(f,t)_\infty}{t^3}\,dt + E_0(f)_\infty\right\}, \tag{1.15}$$

where M is independent of f and n.

In Section 2, we prove Theorem 1. Theorems 2 and 3 are proved in Section 3. Finally, in Section 4, we give an algorithm for singular detection by means of Bernstein-Durrmeyer operators.

2. PROOF OF THEOREM 1

To prove Theorem 1, we need some preliminary results. By simple computations, we have the moments of Bernstein-Durrmeyer operators.

LEMMA 2.1. *Let $D_n(f,x) = D_n(f(t),x)$ be given by (1.1). Then we have for $x \in [0,1]$*

$$\begin{aligned} D_n(1,x) &= 1; \\ D_n(t,x) &= \frac{n}{n+2}x + \frac{1}{n+2}; \\ D_n(t^2,x) &= \frac{n^2x^2 + 3nx + 2 + nx(1-x)}{(n+2)(n+3)}. \end{aligned}$$

Hence,

$$D_n\left((t-x)^2, x\right) \le 2\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right) \tag{2.1}$$

and

$$\left|D_n(\varphi^2(t),x) - \varphi^2(x)\right| \le \frac{3}{n}. \tag{2.2}$$

In the proof of the inverse part, we need some Bernstein-Markov-type inequalities as follows.

LEMMA 2.2. *Let $f \in C[0,1]$, $2 \le n \in \mathbb{N}$. Then we have*

$$D'_n(f,x) = n\sum_{k=0}^{n-1} P_{n-1,k}(x)\left((n+1)\int_0^1 f(t)P_{n,k+1}(t)\,dt - (n+1)\int_0^1 f(t)P_{n,k}(t)\,dt\right), \tag{2.3}$$

which implies

$$\|D'_n(f)\|_\infty \le 2n\|f\|_\infty.$$

Note that $P'_{n,k}(x) = n(P_{n-1,k-1}(x) - P_{n-1,k}(x))$. The proof of Lemma 2.2 is quite easy and we omit it here.

LEMMA 2.3. *Let $2 \le n \in \mathbb{N}$, $0 < \alpha \le 1$, $E \subset [0,1]$, $f \in C[0,1]$ satisfy*

$$|f(t)| \le \left(\frac{t(1-t)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(t,E))^\alpha, \qquad t \in [0,1]. \tag{2.4}$$

Then we have

$$|D'_n(f,x)| \le 6\sqrt{\frac{n}{x(1-x)}}\left\{\left(\frac{x(1-x)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^\alpha\right\}, \qquad x \in [0,1]. \tag{2.5}$$

PROOF OF LEMMA 2.3. We note that for $a, b \ge 0$, $0 < \beta \le 1$

$$(a+b)^\beta \le a^\beta + b^\beta \tag{2.6}$$

and for t, $x \in [0,1]$,

$$d(t,E) \le d(x,E) + |t-x|. \tag{2.7}$$

$$\varphi^2(t) \le \varphi^2(x) + 2|t-x|. \tag{2.8}$$

We also put $P_{n,k} = 0$ for $k < 0$ or $k > n$. Then, we have

$$P'_{n,k}(x) = \frac{k - nx}{x(1-x)} P_{n,k}(x).$$

Using this formula and the above notes, we have from assumption (2.4) that

$$\begin{aligned}
|D_n'(f,x)| &= \left|\sum_{k=0}^{n} \frac{k-nx}{x(1-x)} P_{n,k}(x)(n+1) \int_0^1 f(t)P_{n,k}(t)\,dt\right| \\
&\le \sum_{k=0}^{n} \frac{|k-nx|}{x(1-x)} P_{n,k}(x)(n+1) \\
&\quad \times \int_0^1 P_{n,k}(t) \left\{ \left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2} + \left(\frac{2|t-x|}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} + |t-x|^{\alpha} \right\} dt \\
&\le \sum_{k=0}^{n} \frac{|k-nx|}{x(1-x)} P_{n,k}(x) \left\{ \left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2} + 2n^{-\alpha} + (d(x,E))^{\alpha} \right\} \\
&\quad + 2\sum_{k=0}^{n} \frac{|k-nx|}{x(1-x)} P_{n,k}(x)(n+1) \int_0^1 P_{n,k}(t)|t-x|^{\alpha}\,dt \\
&\le \left\{ \left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2} + 2n^{-\alpha} + (d(x,E))^{\alpha} \right\} \frac{n}{x(1-x)} \left(B_n\left((t-x)^2,x\right)\right)^{1/2} \\
&\quad + \frac{2n}{x(1-x)} \left(B_n\left((t-x)^2,x\right)\right)^{1/2} \left\{ \sum_{k=0}^{n} P_{n,k}(x)(n+1) \int_0^1 P_{n,k}(t)(t-x)^2\,dt \right\}^{\alpha/2} \\
&\le 2\sqrt{\frac{n}{x(1-x)}} \left\{ \left(\frac{x(1-x)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} \right\} \\
&\quad + 2\sqrt{\frac{n}{x(1-x)}} \left\{ 2\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right) \right\}^{\alpha/2} \\
&\le 6\sqrt{\frac{n}{x(1-x)}} \left\{ \left(\frac{x(1-x)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} \right\}.
\end{aligned}$$

Here we have used (2.1) and Hölder's inequality several times, and $B_n(g,x)$ is the Bernstein operator defined on $C[0,1]$ as

$$B_n(g,x) = \sum_{k=0}^{n} g\left(\frac{k}{n}\right) P_{n,k}(x). \tag{2.9}$$

We have also used the moment of the Bernstein operators (cf. [10–12]), namely

$$B_n\left((t-x)^2,x\right) = \frac{x(1-x)}{n}. \tag{2.10}$$

The proof of Lemma 2.3 is complete.

LEMMA 2.4. *Under the same conditions as in Lemma 2.3, we have*

$$|D_n'(f,x)| \le 20n \left\{ \left(\frac{x(1-x)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} \right\}, \qquad x \in [0,1]. \tag{2.11}$$

PROOF OF LEMMA 2.4. By (2.3), (2.6), (2.7), and (2.8), we have

$$\begin{aligned}
|D_n'(f,x)| &\le n \sum_{k=0}^{n-1} P_{n-1,k}(x)(n+1) \int_0^1 \left(P_{n,k+1}(t) + P_{n,k}(t)\right) \\
&\quad \times \left\{ \left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2} + \left(\frac{2|t-x|}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} + |t-x|^{\alpha} \right\} dt
\end{aligned}$$

$$
\begin{aligned}
&\leq 2n\left\{\left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2}+2n^{-\alpha}+(d(x,E))^{\alpha}\right\}+2n\sum_{k=0}^{n-1}P_{n-1,k}(x)(n+1)\\
&\quad\times\left\{\int_0^1(P_{n,k+1}(t)+P_{n,k}(t))(t-x)^2\,dt\right\}^{\alpha/2}\\
&\quad\times\left\{\int_0^1(P_{n,k+1}(t)+P_{n,k}(t))\,dt\right\}^{1-\alpha/2}\\
&\leq 2n\left\{\left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2}+2n^{-\alpha}+(d(x,E))^{\alpha}\right\}\\
&\quad+4n\left\{\sum_{k=0}^{n-1}P_{n-1,k}(x)(n+1)\int_0^1(P_{n,k+1}(t)+P_{n,k}(t))\,(t-x)^2\,dt\right\}^{\alpha/2}. \qquad (2.12)
\end{aligned}
$$

Now we estimate the last terms:

$$
\begin{aligned}
&\sum_{k=0}^{n-1}P_{n-1,k}(x)(n+1)\int_0^1 P_{n,k+1}(t)(t-x)^2\,dt\\
&=\sum_{k=0}^{n-1}P_{n-1,k}(x)(n+1)\left\{\frac{(k+2)(k+3)}{(n+1)(n+2)(n+3)}-2x\,\frac{k+2}{(n+1)(n+2)}+\frac{x^2}{n+1}\right\}\\
&=\sum_{k=0}^{n-1}P_{n-1,k}(x)\left\{\frac{k^2}{(n+2)(n+3)}+\left(\frac{5}{(n+2)(n+3)}-\frac{2x}{n+2}\right)k\right.\\
&\qquad\qquad\left.+\frac{6}{(n+2)(n+3)}-\frac{4x}{n+2}+x^2\right\}\\
&=\frac{13-n}{(n+2)(n+3)}\,x^2+\frac{n-1}{(n+2)(n+3)}\,x(1-x)+\frac{6}{(n+2)(n+3)}+\frac{n-17}{(n+2)(n+3)}\,x\\
&=\frac{2n-1}{(n+2)(n+3)}\,x(1-x)+\frac{13x^2-17x+6}{(n+2)(n+3)}\\
&\leq\frac{2}{n}\,x(1-x)+\frac{6}{n^2}\\
&\leq 6\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right). \qquad (2.13)
\end{aligned}
$$

In the same way, we have

$$
\sum_{k=0}^{n-1}P_{n-1,k}(x)(n+1)\int_0^1 P_{n,k}(t)(t-x)^2\,dt\leq 6\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right). \qquad (2.14)
$$

Combining (2.13) and (2.14), we have from (2.12)

$$
|D_n'(f,x)|\leq 20n\left\{\left(\frac{\varphi^2(x)}{n}\right)^{\alpha/2}+n^{-\alpha}+(d(x,E))^{\alpha}\right\}.
$$

The proof of Lemma 2.4 is complete.

Finally, we shall use the following inequality.

LEMMA 2.5. *Let $0\leq\beta<1$, $0\leq x_1<x_2\leq 1$, $x_2-x_1\leq 1/4$. Then we have*

$$
\int_{x_1}^{x_2}(u(1-u))^{(\beta-1)/2}\,du\leq 4\,\frac{x_2-x_1}{(\max\{\varphi^2(x_2),\varphi^2(x_1)\})^{(1-\beta)/2}}. \qquad (2.15)
$$

PROOF OF LEMMA 2.5. We prove this lemma by discriminating three cases.

Let $0 \le x_1 < x_2 \le 1/2$. Then for $u \in [x_1, x_2]$, $1/2 \le 1-u \le 1$. We have

$$\begin{aligned}\int_{x_1}^{x_2} (u(1-u))^{(\beta-1)/2}\,du &\le 2^{(1-\beta)/2} \frac{1}{(\beta-1)/2+1}\left(x_2^{(\beta+1)/2} - x_1^{(\beta+1)/2}\right)\\ &\le \frac{2}{\beta+1} 2^{(1-\beta)/2} \frac{x_2-x_1}{x_2^{(1-\beta)/2}}\\ &\le \frac{2}{1+\beta} 2^{(1-\beta)/2} \frac{x_2-x_1}{(\max\{x_2,x_1\})^{(1-\beta)/2}}\\ &\le \frac{2}{\beta+1} 2^{(1-\beta)/2} \frac{x_2-x_1}{(\max\{\varphi^2(x_2),\varphi^2(x_1)\})^{(1-\beta)/2}}.\end{aligned}$$

If $x_1 < 1/2 < x_2$, then by $x_2 - x_1 \le 1/4$, we know that $1/4 < x_1 < 1/2 < x_2 < 3/4$. Hence, for $u \in [x_1, x_2]$, $3/16 \le \varphi^2(u) \le 1/4$ and

$$\begin{aligned}\int_{x_1}^{x_2} (u(1-u))^{(\beta-1)/2}\,du &\le \left(\frac{16}{3}\right)^{(1-\beta)/2} (x_2-x_1)\\ &\le \left(\frac{4}{3}\right)^{(1-\beta)/2} \frac{x_2-x_1}{(\max\{\varphi^2(x_2),\varphi^2(x_1)\})^{(1-\beta)/2}}.\end{aligned}$$

Finally, let $1/2 \le x_1 < x_2 \le 1$. Then we replace u by $1-u$, reduce this case to the first case and obtain

$$\begin{aligned}\int_{x_1}^{x_2} (u(1-u))^{(\beta-1)/2}\,du &= \int_{1-x_2}^{1-x_1} (u(1-u))^{(\beta-1)/2}\,du\\ &\le \frac{2}{\beta+1} 2^{(1-\beta)/2} \frac{x_2-x_1}{(\max\{\varphi^2(x_2),\varphi^2(x_1)\})^{(1-\beta)/2}}.\end{aligned}$$

Combining the above three cases, we have for $0 \le x_1 < x_2 \le 1$, $x_2 - x_1 \le 1/4$,

$$\int_{x_1}^{x_2} (u(1-u))^{(\beta-1)/2}\,du \le 2\sqrt{2}\,\frac{x_2-x_1}{(\max\{\varphi^2(x_2),\varphi^2(x_1)\})^{(1-\beta)/2}}.$$

Hence (2.15) holds and the proof of Lemma 2.5 is complete.

After the above preparation, we can now prove Theorem 1.

PROOF OF THEOREM 1. We first prove the direct part. Suppose that (1.6) holds. We remark that (1.6) also holds for $x \in [0,1]$ and $y \in \overline{E}$, the closure of the set E.

Now we prove (1.7). Let $n \in \mathbb{N}$, $x \in [0,1]$ and $x_0 \in \overline{E}$ be such that

$$|x - x_0| = d(x, E) = d\left(x, \overline{E}\right).$$

Then by Lemma 2.1, we have

$$\begin{aligned}|D_n(f,x) - f(x)| &\le D_n(|f(t) - f(x_0)|, x) + |f(x) - f(x_0)|\\ &\le D_n\left(M_f|t-x_0|^\alpha, x\right) + M_f|x-x_0|^\alpha\\ &\le M_f\left\{D_n\left(|t-x|^\alpha, x\right) + |x-x_0|^\alpha\right\} + M_f|x-x_0|^\alpha\\ &\le M_f\left\{D_n\left((t-x)^2, x\right)\right\}^{\alpha/2} + 2M_f|x-x_0|^\alpha\\ &\le 2M_f\left(\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\alpha/2} + (d(x,E))^\alpha\right).\end{aligned}$$

Hence (1.7) holds and our proof of the direct part is complete.

We note that this direct part holds true also for $\alpha = 1$. Now we prove the inverse part. Here the commutativity of the Bernstein-Durrmeyer operators is crucial:

$$D_n(D_m f) = D_m(D_n f), \qquad m, n \in \mathbb{N}. \tag{2.16}$$

Suppose that (1.7) holds. Let $x \in [0,1]$ and $y \in E$. We want to prove (1.6). If $|x-y| \geq 1/4$, then we can easily obtain

$$|f(x) - f(y)| \leq 2\|f\|_\infty \leq 8\|f\|_\infty |x-y|^\alpha.$$

If $|x-y| < 1/4$, we choose $5 \leq n \in \mathbb{N}$ such that

$$\frac{|x-y|}{2} < \max\left\{\frac{1}{2^{n-2}}, \frac{\varphi(x)}{\sqrt{2^{n-2}}}, \frac{\varphi(y)}{\sqrt{2^{n-2}}}\right\} \leq |x-y|. \tag{2.17}$$

This is possible because the sequence

$$\left\{\delta(n,x,y) := \max\left\{\frac{1}{(2^{n-2})}, \frac{\varphi(x)}{\sqrt{2^{n-2}}}, \frac{\varphi(y)}{\sqrt{2^{n-2}}}\right\}\right\}_{n\in\mathbb{N}}$$

decreases monotonically to zero as n tends to infinity, and since it satisfies

$$\delta(n,x,y) < \delta(n-1,x,y) \leq 2\delta(n,x,y), \qquad n \in \mathbb{N}. \tag{2.18}$$

Under this choice, using (2.6), (2.7), (2.8), (2.16), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - D_{2^n}(f,x)| + |D_{2^n}(f,y) - f(y)| \\
&\quad + |D_{2^n}(f - D_{2^{n-1}}(f), x) - D_{2^n}(f - D_{2^{n-1}}(f), y)| + |D_{2^n}(D_{2^{n-1}}f, x) - D_{2^n}(D_{2^{n-1}}f, y)| \\
&\leq M_f'\left(\left(\frac{\varphi^2(x)}{2^n}\right)^{\alpha/2} + \left(2^{-2n}\right)^{\alpha/2} + (d(x,E))^\alpha\right) \\
&\quad + M_f'\left(\left(\frac{\varphi^2(y)}{2^n}\right)^{\alpha/2} + \left(2^{-2n}\right)^{\alpha/2} + (d(y,E))^\alpha\right) \\
&\quad + M_f' D_{2^n}\left(\left(\frac{t(1-t)}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^\alpha + (d(t,E))^\alpha, x\right) \\
&\quad + M_f' D_{2^n}\left(\left(\frac{t(1-t)}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^\alpha + (d(t,E))^\alpha, y\right) \\
&\quad + \left|\sum_{j=2}^{n} \{[D_{2^j}(D_{2^{j-1}}f)(x) - D_{2^{j-1}}(D_{2^{j-2}}f)(x)] - [D_{2^j}(D_{2^{j-1}}f)(y) - D_{2^{j-1}}(D_{2^{j-2}}f)(y)]\}\right| \\
&\quad + |D_2(D_1 f)(x) - D_2(D_1 f)(y)| \\
&\leq M_f'\left(5|x-y|^\alpha + 2(d(x,E))^\alpha\right) \\
&\quad + M_f' D_{2^n}\left(\left(\frac{\varphi^2(x)}{2^{n-1}}\right)^{\alpha/2} + \left(\frac{2|t-x|}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^\alpha + (d(x,E))^\alpha + |t-x|^\alpha, x\right) \\
&\quad + M_f' D_{2^n}\left(\left(\frac{\varphi^2(y)}{2^{n-1}}\right)^{\alpha/2} + \left(\frac{2|t-y|}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^\alpha + |t-y|^\alpha, y\right) \\
&\quad + \sum_{j=2}^{n} \left|\int_x^y \left| D_{2^{j-1}}'(D_{2^j}f - D_{2^{j-2}}f, t)\right| dt\right| + 4\|f\|_\infty |x-y|
\end{aligned}$$

$$\leq M'_f \left\{ 11|x-y|^\alpha + 3(d(x,E))^\alpha + 2(D_{2^n}\left((t-x)^2,x\right))^{\alpha/2} + 2\left(D_{2^n}((t-y)^2,y)\right)^{\alpha/2} \right\}$$

$$+ 4\|f\|_\infty |x-y|^\alpha + \sum_{j=2}^{n} \left| \int_x^y \right| D'_{2^{j-1}}\left(D_{2^j} f - D_{2^{j-2}} f, t\right) |dt|$$

$$\leq \left(30M'_f + 4\|f\|_\infty\right)|x-y|^\alpha + \sum_{j=2}^{n} I_j. \tag{2.19}$$

Here we have used the fact that $d(x,E) \leq |x-y|$ and Hölder's inequality. We have also used the abbreviation

$$I_j = \left| \int_x^y \right| D'_{2^{j-1}}(D_{2^j} f - D_{2^{j-2}} f, t)\, |dt|. \tag{2.20}$$

Next we estimate $\sum_{j=2}^{n} I_j$. We separate the estimates into three cases.

The first case we want to consider is $\delta(n,x,y) = \max\{1/2^{n-2}, \varphi(x)/\sqrt{2^{n-2}}, \varphi(y)/\sqrt{2^{n-2}}\} = 1/2^{n-2}$; i.e., $\max\{\varphi(x)/\sqrt{2^{n-2}}, \varphi(y)/\sqrt{2^{n-2}}\} \leq 1/2^{n-2}$. Hence $2^{n-2} < 2/|x-y|$ by (2.17). We note that for $t \in [x,y]$ or $[y,x]$,

$$d(t,E) \leq |t-y| \leq |x-y|. \tag{2.21}$$

Then we can use Lemma 2.4 for 2^{j-1} and obtain

$$I_j \leq \left| \int_x^y 60M'_f 2^{j-1} \left\{ \left(\frac{t(1-t)}{2^{j-1}}\right)^{\alpha/2} + (2^{j-1})^{-\alpha} + (d(t,E))^\alpha \right\} dt \right|$$

$$\leq 60M'_f \left\{ |x-y|^{\alpha+1} 2^{j-1} + \left(2^{j-1}\right)^{1-\alpha} |x-y| + \left(2^{j-1}\right)^{1-\alpha/2} \left| \int_x^y (t(1-t))^{\alpha/2}\, dt \right| \right\}.$$

Taking sums over $2 \leq j \leq n$, we have

$$\sum_{j=2}^{n} I_j \leq 60M'_f \left\{ 2^n|x-y|^{\alpha+1} + \frac{1}{1-2^{\alpha-1}}\,(2^{n-1})^{1-\alpha}|x-y| \right.$$

$$\left. + \left| \int_x^y (t(1-t))^{\alpha/2}\, dt \right| \frac{1}{1-2^{\alpha/2-1}}\,(2^{n-1})^{1-\alpha/2} \right\}. \tag{2.22}$$

We state that for $0 < \alpha \leq 1$, $0 \leq x_1 < x_2 \leq 1$, $x_2 - x_1 < 1/4$,

$$\int_{x_1}^{x_2} (t(1-t))^{\alpha/2}\, dt \leq \left(\frac{2}{\sqrt{3}}\right)^\alpha (\max\{\varphi(x_1), \varphi(x_2)\})^\alpha (x_2 - x_1). \tag{2.23}$$

We now prove inequality (2.23). Let $0 \leq x_1 < x_2 \leq 1/2$ or $1/2 \leq x_1 < x_2 \leq 1$. Then $\varphi^2(t)$ is monotone in $[x_1, x_2]$. Hence,

$$\int_{x_1}^{x_2} (t(1-t))^{\alpha/2} dt \leq (\max\{\varphi(x_1), \varphi(x_2)\})^\alpha (x_2 - x_1).$$

If $x_1 < 1/2 < x_2$, then $1/4 < x_1 < 1/2 < x_2 < 3/4$. Hence, $\max\{\varphi(x_1), \varphi(x_2)\} \geq \varphi(1/4) = \sqrt{3}/4$. Therefore, we have

$$\int_{x_1}^{x_2} (t(1-t))^{\alpha/2}\, dt \leq 4^{-\alpha/2}(x_2 - x_1) \leq \left(\frac{2}{\sqrt{3}}\right)^\alpha (\max\{\varphi(x_1), \varphi(x_2)\})^\alpha (x_2 - x_1).$$

Thus, statement (2.23) is valid.

Using the inequality (2.23) in (2.22), we obtain

$$\begin{aligned}
\sum_{j=2}^{n} I_j &\le 60M_f' \left\{ 4\frac{2}{|x-y|}|x-y|^{1+\alpha} + \frac{2^{1-\alpha}}{1-2^{\alpha-1}}\left(\frac{2}{|x-y|}\right)^{1-\alpha}|x-y| \right. \\
&\quad \left. + \frac{1}{1-2^{\alpha/2-1}}\left(\frac{2}{\sqrt{3}}\right)^{\alpha}|x-y|\left(\max\left\{\frac{\varphi(x)}{\sqrt{2^{n-2}}},\frac{\varphi(y)}{\sqrt{2^{n-2}}}\right\}\right)^{\alpha}\frac{2}{|x-y|}\right\} \\
&\le 60M_f'\left(8+\frac{2^{2-\alpha}}{1-2^{\alpha-1}}+\frac{2^{1+\alpha}}{1-2^{\alpha/2-1}}\right)|x-y|^{\alpha}. \qquad (2.24)
\end{aligned}$$

Thus, we have estimated $\sum_{j=2}^{n} I_j$ when $\delta(n,x,y) = 1/2^{n-2}$.

On the other hand, when

$$\delta(n,x,y) = \max\left\{\frac{\varphi(x)}{\sqrt{2^{n-2}}},\frac{\varphi(y)}{\sqrt{2^{n-2}}}\right\},$$

we have $\max\{\varphi(x),\varphi(y)\} \ge 2^{(2-n)/2}$, and hence

$$\frac{|x-y|}{2} < \max\left\{\frac{\varphi(x)}{\sqrt{2^{n-2}}},\frac{\varphi(y)}{\sqrt{2^{n-2}}}\right\} \le |x-y|. \qquad (2.25)$$

The second case of our estimates is $0<\alpha<1/2$ and $\delta(n,x,y)=\max\{\varphi(x)/\sqrt{2^{n-2}},\varphi(y)/\sqrt{2^{n-2}}\}$. Here we can use Lemma 2.5 and the estimates are easy. In fact, by Lemma 2.3, Lemma 2.5 and (2.21) we have, for $2 \le j \le n$,

$$\begin{aligned}
I_j &= \left|\int_x^y \left| D'_{2^{j-1}}(D_{2^j}f - D_{2^{j-2}}f,t)\right| dt\right| \\
&\le \left|\int_x^y 18M_f'\sqrt{\frac{2^{j-1}}{t(1-t)}}\left\{\left(\frac{t(1-t)}{2^{j-1}}\right)^{\alpha/2} + \left(2^{1-j}\right)^{\alpha} + (d(t,E))^{\alpha}\right\} dt\right| \\
&\le 18M_f'\left\{(2^{j-1})^{(1-\alpha)/2}\left|\int_x^y (t(1-t))^{(\alpha-1)/2}\,dt\right| \right. \\
&\quad \left. +\sqrt{2^{j-1}}\left(\left(2^{1-j}\right)^{\alpha}+|x-y|^{\alpha}\right)\left|\int_x^y (t(1-t))^{-1/2}\,dt\right|\right\} \\
&\le 18M_f'\left\{4|x-y|(\max\{\varphi(x),\varphi(y)\})^{\alpha-1}(2^{j-1})^{(1-\alpha)/2}\right. \\
&\qquad + 4|x-y|(\max\{\varphi(x),\varphi(y)\})^{-1}\left(2^{j-1}\right)^{1/2-\alpha} \\
&\qquad \left. +4|x-y|^{1+\alpha}(\max\{\varphi(x),\varphi(y)\})^{-1}(2^{j-1})^{1/2}\right\}.
\end{aligned}$$

Thus, taking sums over j, we get from (2.25)

$$\begin{aligned}
\sum_{j=2}^{n} I_j &\le 72M_f'\left\{|x-y|(\max\{\varphi(x),\varphi(y)\})^{\alpha-1}(2^{n-1})^{(1-\alpha)/2}\frac{1}{1-2^{(\alpha-1)/2}}\right. \\
&\quad + |x-y|(\max\{\varphi(x),\varphi(y)\})^{-1}(2^{n-1})^{1/2-\alpha}\frac{1}{1-2^{\alpha-1/2}} \\
&\quad \left. +|x-y|^{1+\alpha}(\max\{\varphi(x),\varphi(y)\})^{-1}(2^{n-1})^{1/2}\frac{1}{1-1/\sqrt{2}}\right\} \\
&\le 72M_f'\left\{\frac{2^{(1-\alpha)/2}}{1-2^{(\alpha-1)/2}}|x-y|\left(\max\left\{\frac{\varphi(x)}{2^{(n-2)/2}},\frac{\varphi(y)}{2^{(n-2)/2}}\right\}\right)^{\alpha-1}\right. \\
&\quad + \frac{2^{1/2-\alpha}}{1-2^{\alpha-1/2}}|x-y|\left(\max\left\{\frac{\varphi(x)}{2^{(n-2)/2}},\frac{\varphi(y)}{2^{(n-2)/2}}\right\}\right)^{-1}(2^{2-n})^{\alpha}
\end{aligned}$$

$$+\frac{\sqrt{2}}{1-2^{-1/2}}|x-y|^{1+\alpha}\left(\max\left\{\frac{\varphi(x)}{2^{(n-2)/2}},\frac{\varphi(y)}{2^{(n-2)/2}}\right\}\right)^{-1}\Bigg\}$$
$$\le 72M_f'\left\{\frac{2^{(1-\alpha)/2}}{1-2^{(\alpha-1)/2}}2^{1-\alpha}|x-y|^\alpha+\frac{2^{3/2-\alpha}}{1-2^{\alpha-1/2}}|x-y|^\alpha+\frac{2^{3/2}}{1-2^{-1/2}}|x-y|^\alpha\right\}$$
$$\le 72M_f'\left(\frac{4^{1-\alpha}}{1-2^{(\alpha-1)/2}}+\frac{2^{2-\alpha}}{1-2^{\alpha-1/2}}+\frac{4}{\sqrt{2}-1}\right)|x-y|^\alpha. \tag{2.26}$$

We have thus completed the estimate of $\sum_{j=2}^n I_j$ in the second case.

The final case is $1/2\le\alpha<1$ and $\delta(n,x,y)=\max\{\varphi(x)/\sqrt{2^{n-2}},\varphi(y)/\sqrt{2^{n-2}}\}$. The estimate in this case is somewhat difficult. We need to combine the inequalities (2.5) and (2.11) here.

Let $2\le j\le n$. If $\max\{\varphi(x),\varphi(y)\}\ge 2^{(2-j)/2}$, i.e., $2^{(j-2)/2}\max\{\varphi(x),\varphi(y)\}\ge 1$. Then by Lemma 2.3, Lemma 2.5, (2.6), (2.7), and (2.8), we have

$$\begin{aligned}
I_j &\le \left|\int_x^y 18M_f'\sqrt{\frac{2^{j-1}}{t(1-t)}}\left\{\left(\frac{t(1-t)}{2^{j-1}}\right)^{\alpha/2}+\left(2^{1-j}\right)^\alpha+(d(t,E))^\alpha\right\}dt\right|\\
&\le 18M_f'\Big\{4(2^{j-1})^{(1-\alpha)/2}|x-y|(\max\{\varphi(x),\varphi(y)\})^{\alpha-1}+4(2^{j-1})^{1/2-\alpha}|x-y|\\
&\qquad\times(\max\{\varphi(x),\varphi(y)\})^{-1}+4|x-y|^\alpha 2^{(j-1)/2}|x-y|(\max\{\varphi(x),\varphi(y)\})^{-1}\Big\}\\
&\le 144M_f'|x-y|^{1+\alpha}\frac{2^{(j-2)/2}}{\max\{\varphi(x),\varphi(y)\}}+288M_f'|x-y|\frac{(2^{j-2})^{(1-\alpha)/2}}{(\max\{\varphi(x),\varphi(y)\})^{1-\alpha}}.
\end{aligned}\tag{2.27}$$

On the other hand, let $\max\{\varphi(x),\varphi(y)\}<2^{(2-j)/2}$, i.e., $2^{j-2}<(2^{(j-2)/2})/(\max\{\varphi(x),\varphi(y)\})$. Then by Lemma 2.4, (2.6), (2.7), (2.8), and (2.23), we have

$$\begin{aligned}
I_j &\le \left|\int_x^y 60M_f'2^{j-1}\left\{\left(\frac{t(1-t)}{2^{j-1}}\right)^{\alpha/2}+\left(2^{1-j}\right)^\alpha+(d(t,E))^\alpha\right\}dt\right|\\
&\le 60M_f'\left(\frac{2}{\sqrt{3}}\right)^\alpha(2^{j-1})^{1-\alpha/2}(\max\{\varphi(x),\varphi(y)\})^\alpha|x-y|\\
&\quad+60M_f'|x-y|(2^{j-1})^{1-\alpha}+60M_f'|x-y|^{1+\alpha}2^{j-1}\\
&\le 240M_f'|x-y|(2^{j-2})^{(1-\alpha)/2}(\max\{\varphi(x),\varphi(y)\})^{\alpha-1}\\
&\quad+120M_f'|x-y|^{1+\alpha}\left(2^{j-2}\right)^{1/2}(\max\{\varphi(x),\varphi(y)\})^{-1}.
\end{aligned}\tag{2.28}$$

Combining (2.27) and (2.28), we have for $2\le j\le n$

$$\begin{aligned}
I_j &\le 144M_f'|x-y|^{1+\alpha}\left(2^{j-2}\right)^{1/2}(\max\{\varphi(x),\varphi(y)\})^{-1}\\
&\quad+288M_f'|x-y|\left(2^{j-2}\right)^{(1-\alpha)/2}(\max\{\varphi(x),\varphi(y)\})^{\alpha-1}.
\end{aligned}$$

By taking sums over j, we have in the final case

$$\begin{aligned}
\sum_{j=2}^n I_j &\le \frac{144\sqrt{2}}{\sqrt{2}-1}M_f'|x-y|^{1+\alpha}\left(\max\left\{\frac{\varphi(x)}{2^{(n-2)/2}},\frac{\varphi(y)}{2^{(n-2)/2}}\right\}\right)^{-1}\\
&\quad+\frac{288}{1-2^{(\alpha-1)/2}}M_f'|x-y|\left(\max\left\{\frac{\varphi(x)}{2^{(n-2)/2}},\frac{\varphi(y)}{2^{(n-2)/2}}\right\}\right)^{\alpha-1}\\
&\le 288\left(\frac{\sqrt{2}}{\sqrt{2}-1}+\frac{2^{1-\alpha}}{1-2^{(\alpha-1)/2}}\right)M_f'|x-y|^\alpha.
\end{aligned}\tag{2.29}$$

Thus, combining (2.24), (2.26), and (2.29), we have in all three cases

$$\sum_{j=2}^n I_j\le C_\alpha M_f'|x-y|^\alpha,$$

where C_α is a constant depending only on α.

Using this estimate in (2.19), we obtain for any $x \in [0,1], y \in E$,

$$|f(x) - f(y)| \le ((30 + C_\alpha)M'_f + 8\|f\|_\infty)|x-y|^\alpha.$$

Therefore, (1.6) holds with $M_f = (30 + C_\alpha)M'_f + 8\|f\|_\infty$.

The proof of Theorem 1 is complete.

3. PROOFS OF THEOREMS 2 AND 3

The proof of Theorem 2 is based on Theorem 3. So we shall first prove Theorem 3. To this end, we need some lemmas.

Let $f \in C[0,1], n \in \mathbb{N} \cup \{0\}$. We denote by $P_n(f)$ the best n^{th} degree algebraic polynomial approximation to f in $C[0,1]$ and write

$$E_n(f)_\infty = \|f - P_n(f)\|_\infty. \tag{3.1}$$

Then we have the following two lemmas which can be found in the monograph of Ditzian and Totik [11].

LEMMA 3.1. *For $\varphi(x) = \sqrt{x(1-x)}$ and $n > 2$*

$$E_n(f)_\infty \le M_1 \omega_\varphi^2 \left(f, \frac{1}{n}\right)_\infty, \tag{3.2}$$

where M_1 is a constant independent of $n > 2$ and $f \in C[0,1]$.

LEMMA 3.2. *For $f \in C[0,1], \varphi(x) = \sqrt{x(1-x)}$ and $n \in \mathbb{N}$*

$$\|\varphi^2 P_n''(f)\|_\infty \le M_2 n^2 \omega_\varphi^2 \left(f, \frac{1}{n}\right)_\infty, \tag{3.3}$$

where M_2 is a constant independent of n and f.

The following Markov inequality is also well known (cf. [12]).

LEMMA 3.3. *Let $n \in \mathbb{N}$ and P_n be an n^{th} degree algebraic polynomial. We have*

$$\|P_n'\|_{C[0,1]} \le n^2 \|P_n\|_{C[0,1]}. \tag{3.4}$$

Finally, we shall also use the following direct result which can be found in D.-X. Zhou [13]. For completeness, we give a somewhat detailed form.

LEMMA 3.4. *Let $f \in C^2[0,1]$, $n \in \mathbb{N}$. Then we have*

$$\|D_n(f) - f\|_\infty \le \frac{4}{n}\left(\|f'\|_\infty + \|\varphi^2 f''\|_\infty\right). \tag{3.5}$$

PROOF OF LEMMA 3.4. Let $x \in (0,1)$ and $\varphi^2(x) < 1/n$. Then, by (2.1), the Taylor-type expansion

$$f(t) = f(x) + \int_x^t f'(u)\,du,$$

and Hölder's inequality, we have

$$\begin{aligned}
|D_n(f,x) - f(x)| &= \left|D_n\left(\int_x^t f'(u)\,du, x\right)\right| \\
&\le \|f'\|_\infty \sqrt{D_n((t-x)^2, x)} \\
&\le \frac{2}{n}\|f'\|_\infty.
\end{aligned}$$

Let $\varphi^2(x) \geq 1/n$. Now we use the Taylor type expansion

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)\,du$$

and Lemma 2.1. Then we obtain

$$\begin{aligned}
|D_n(f,x) - f(x)| &= \left| f'(x)D_n(t-x,x) + D_n\left(\int_x^t (t-u)f''(u)\,du, x\right)\right| \\
&\leq \left|\frac{1-2x}{n+2}\right| \|f'\|_\infty + D_n\left((t-x)^2, x\right)\frac{\|\varphi^2 f''\|_\infty}{x(1-x)} \\
&\leq \frac{4}{n}\left(\|f'\|_\infty + \|\varphi^2 f''\|_\infty\right).
\end{aligned}$$

Here we have used the following inequality:

$$\left|\frac{t-u}{u(1-u)}\right| \leq \frac{|t-x|}{x(1-x)}, \qquad u \in [x,t] \text{ or } [t,x]. \tag{3.6}$$

Thus, for $x \in (0,1)$, we have

$$|D_n(f,x) - f(x)| \leq \frac{4}{n}\left(\|f'\|_\infty + \|\varphi^2 f''\|_\infty\right).$$

The proof of Lemma 3.4 is complete.

With all the above lemmas, we can now prove Theorem 3.

PROOF OF THEOREM 3. We can assume $P_0(f) = 0$, otherwise we only need to replace f by $f - P_0(f)$. Under this assumption, $E_0(f)_\infty = \|f\|_\infty$. Then for $n \leq 3$, (1.11) holds:

$$\|D_n(f) - f\|_\infty \leq 2\|f\|_\infty \leq \frac{6E_0(f)_\infty}{n} \leq \frac{6}{n}\left\{\int_{1/\sqrt{n}}^{1/2} \frac{\omega_\varphi^2(f,t)_\infty}{t^3}\,dt + E_0(f)_\infty\right\}.$$

For $n \in \mathbb{N}$, $n \geq 4$, we let $2 \leq m \in \mathbb{N}$ be such that

$$2^{m-1} \leq \sqrt{n} < 2^m.$$

With this choice, we use Lemmas 3.1–3.4 to obtain

$$\begin{aligned}
&\|D_n(f) - f\|_\infty \leq \|D_n(f - P_{2^m}(f))\|_\infty + \|f - P_{2^m}(f)\|_\infty + \|D_n(P_{2^m}(f)) - P_{2^m}(f)\|_\infty \\
&\quad\leq 2\|f - P_{2^m}(f)\|_\infty + \frac{4}{n}\left(\|P'_{2^m}(f)\|_\infty + \|\varphi^2 P''_{2^m}(f)\|_\infty\right) \\
&\quad\leq 2E_{2^m}(f)_\infty + \frac{4}{n}\left(M_2(2^m)^2\omega_\varphi^2(f,2^{-m})_\infty + \sum_{j=1}^m \|P'_{2^j}(f) - P'_{2^{j-1}}(f)\|_\infty + \|P'_1(f) - P'_0(f)\|_\infty\right) \\
&\quad\leq 2M_1\omega_\varphi^2(f,2^{-m})_\infty \\
&\qquad+ \frac{4}{n}\left\{M_2 2^{2m}\omega_\varphi^2(f,2^{-m})_\infty + \|P_1(f) - P_0(f)\|_\infty + \sum_{j=1}^m \left(\|P_{2^j}(f) - P_{2^{j-1}}(f)\|_\infty 2^{2j}\right)\right\} \\
&\quad\leq (2M_1 + 16M_2)\omega_\varphi^2(f,2^{-m})_\infty + \frac{8E_0(f)_\infty}{n} + \left\{\sum_{j=1}^m \left(2^{2j+1}E_{2^{j-1}}(f)_\infty\right)\right\}\frac{4}{n} \\
&\quad\leq (2M_1 + 16M_2)\omega_\varphi^2(f,2^{-m})_\infty + \frac{168E_0(f)_\infty}{n} + \frac{8M_1}{n}\sum_{j=3}^m \left\{2^{2j}\omega_\varphi^2(f,2^{1-j})_\infty\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{16M_1 + 64M_2}{n} \sum_{j=3}^{m} \left\{2^{2j}\omega_\varphi^2\left(f, 2^{1-j}\right)_\infty\right\} + \frac{168E_0(f)_\infty}{n} \\
&\le \frac{16M_1 + 64M_2}{n} \sum_{j=3}^{m} \left\{\frac{4}{\ln 2}\int_{2^{1-j}}^{2^{2-j}} \frac{\omega_\varphi^2(f,t)_\infty}{t^3}\,dt\right\} + \frac{168E_0(f)_\infty}{n} \\
&\le \frac{M}{n}\left\{\int_{1/\sqrt{n}}^{1/2} \frac{\omega_\varphi^2(f,t)_\infty}{t^3}\,dt + E_0(f)_\infty\right\}.
\end{aligned}
$$

Here $M := (64M_1 + 256M_2)/\ln 2 + 168$ is a constant independent of f and n. Hence, the proof of Theorem 3 is complete.

By means of this direct estimate, we can now prove Theorem 2.

PROOF OF THEOREM 2. Suppose that E is a nonempty subset of $[0,1]$, and that $\overline{E}$ does not contain one of the endpoints of the interval $[0,1]$. We want to find a function $f \in C[0,1]$ such that for $\alpha = 1$, (1.7) holds while (1.6) does not hold.

To this end, we choose a point $x_0 \in E$ and let

$$f(x) = (x - x_0)\ln|x - x_0| \in C[0,1]. \tag{3.7}$$

It is well known (cf. [12,14]) that f is in the Zygmund class, i.e.,

$$\left|\Delta_h^2 f(x)\right| \le M_f|h|,$$

where M_f is independent of h and x. Hence,

$$\omega_\varphi^2(f,t)_\infty \le M_f t, \qquad t > 0.$$

By Theorem 3, we know that

$$\|D_n(f) - f\|_\infty \le M(M_f + \|f\|_\infty)n^{-1/2}. \tag{3.8}$$

We now prove that f satisfies (1.7). Note that $\{0,1\} \cap \overline{E} = \phi$. We let

$$C_1 = \inf\{x : x \in E\}, \qquad C_2 = \sup\{x : x \in E\}.$$

Then $0 < C_1 \le C_2 < 1$. Thus, for $x \in [0,1]$,

$$\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{1/2} + d(x,E) \ge \frac{\sqrt{x(1-x)} + d(x,E)}{\sqrt{n}}. \tag{3.9}$$

We state that for $x \in [0,1]$

$$\sqrt{x(1-x)} + d(x,E) \ge \min\left\{\frac{\sqrt{C_1(1-C_2)}}{2}, \frac{C_1}{2}, \frac{1-C_2}{2}\right\} > 0. \tag{3.10}$$

In fact, when $x \in [C_1/2, (1+C_2)/2]$,

$$\sqrt{x(1-x)} \ge \frac{\sqrt{C_1(1-C_2)}}{2}$$

and when $x \notin [C_1/2, (1+C_2)/2]$,

$$d(x,E) \ge \min\left\{\frac{C_1}{2}, \frac{1-C_2}{2}\right\}.$$

Therefore, (3.10) holds and

$$\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right)^{1/2}+d(x,E)\geq\frac{C_0}{\sqrt{n}},\qquad x\in[0,1], \tag{3.11}$$

where $C_0>0$ is a positive constant independent of n and x.

Combining (3.8) and (3.11), we know that f satisfies (1.7):

$$\begin{aligned}|D_n(f,x)-f(x)|&\leq\|D_n(f)-f\|_\infty\leq M\left(M_f+\|f\|_\infty\right)n^{-1/2}\\&\leq\frac{M}{C_0}\left(M_f+\|f\|_\infty\right)\left\{\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right)^{1/2}+d(x,E)\right\},\qquad x\in[0,1].\end{aligned}$$

It is trivial that f is not locally Lip 1 at the point $x_0\in E$. Hence, (1.6) does not hold for f and $\alpha=1$.

The proof of Theorem 2 is complete.

REMARK. We do not know whether Theorem 2 holds when $\overline{E}$ contains one endpoint.

4. AN APPLICATION IN SINGULAR DETECTION

In this section, we give an algorithm for singular detection by means of Bernstein-Durrmeyer operators.

We say that $x_0\in[0,1]$ is a singular point of a continuous signal $f\in C[0,1]$ if f is not locally Lip 1 at x_0. We denote $S(f)$ as the set of all the singular points of f. For $0<\alpha<1$, we denote $S_\alpha(f)$ as the set of all the singular points with Hölder exponent α. It is evident that $\bigcup_{0<\alpha<1}S_\alpha(f)\subset S(f)$.

We let

$$S_1(f)=\{y\in[0,1]:\ (1.7)\text{ is not satisfied for }E=\{y\},\alpha=1\text{ and any constant }M_f'>0\}. \tag{4.1}$$

Then we know that

$$\bigcup_{0<\alpha<1}S_\alpha(f)\subset S_1(f)\subset S(f). \tag{4.2}$$

By Theorem 2, it may happen that $S_1(f)\neq S(f)$. It may also happen that $\bigcup_{0<\alpha<1}S_\alpha(f)\neq S_1(f)$, which can be seen from the following example.

EXAMPLE. Let $f(x)=x\ln(x)-x\in C[0,1]$. Then f is locally Lip α at 0 for any $0<\alpha<1$. Hence $0\notin\bigcup_{0<\alpha<1}S_\alpha(f)$. On the other hand, we state that $0\in S_1(f)$, i.e., (1.7) does not hold for $E=\{0\}$ and $\alpha=1$.

To prove this statement, we suppose on the contrary that

$$|D_n(f,x)-f(x)|\leq M_f'\left(\sqrt{\frac{x(1-x)}{n}}+\frac{1}{n}+x\right)\leq 2M_f'\left(\frac{1}{n}+x\right),\qquad x\in[0,1]. \tag{4.3}$$

Then we have for $x\in(0,1)$

$$\begin{aligned}|D_n(f,x)-f(x)|&=\left|f'(x)D_n(t-x,x)+D_n\left(\int_x^t(t-u)f''(u)\,du,x\right)\right|\\&\geq|f'(x)|\frac{|1-2x|}{n+2}-\|\varphi^2f''\|_\infty D_n\left(\frac{(t-x)^2}{x(1-x)},x\right)\\&\geq|f'(x)|\frac{|1-2x|}{n+2}-\frac{2}{x(1-x)}\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right).\end{aligned}$$

Let $x = 1/n$. Then we obtain

$$\left|f'\left(\frac{1}{n}\right)\right| \frac{1-2/n}{n+2} \le \frac{4M'_f}{n} + \frac{2}{n} + \frac{2}{n-1},$$

and for $n \ge 3$

$$\left|f'\left(\frac{1}{n}\right)\right| = \ln(n) \le \frac{n(n+2)}{n-2} \left\{\frac{2+4M'_f}{n} + \frac{2}{n-1}\right\} \le 30 + 20M'_f,$$

which is a contradiction. Therefore, $0 \in S_1(f)$ and hence $S_1(f) \ne \bigcup_{0<\alpha<1} S_\alpha(f)$.

Our algorithm will give a method to determine the set $S_1(f)$ which satisfies (4.2). To this end, we write $S_1(f)$ as

$$S_1(f) = \bigcap_{m\in\mathbb{N}} F_m, \tag{4.4}$$

where

$$\begin{aligned} F_m &= \bigcup_{n\in\mathbb{N}} F_{n,m}, \\ F_{n,m} &= \bigcup_{x\in[0,1]} F_{n,m,x}, \\ F_{n,m,x} &= \left\{y \in [0,1] : \frac{|D_n(f,x) - f(x)|}{\varphi(x)/\sqrt{n} + 1/n + |x-y|} > m\right\} \\ &= \left(x + \frac{1}{n} + \frac{\varphi(x)}{\sqrt{n}} - \frac{|D_n(f,x)-f(x)|}{m}, x - \frac{1}{n} - \frac{\varphi(x)}{\sqrt{n}} + \frac{|D_n(f,x)-f(x)|}{m}\right) \cap [0,1]. \end{aligned} \tag{4.5}$$

Here we set $(a,b) = \phi$ if $b \le a$.

Now we can describe our algorithm as follows.

ALGORITHM. Let $f \in C[0,1]$. Then the set $S_1(f)$ of singular points can be computed approximately by the following steps:

1. Choose $p, q, l \in \mathbb{N}$, and let $x_j = j2^{-l}$, $j = 0, 1, \dots, 2^l$.
2. Compute $\{\tilde{D}_{n,l}(f, x_j) : 0 \le j \le 2^l, 1 \le n \le 2^q\}$ by the product of the matrix A_n and the vector $\tilde{f} = (f(x_0), \dots, f(x_{2^l}))$

$$\tilde{D}_{n,l}(f, x_j) = (A_n\tilde{f})_j = \sum_{k=0}^{2^l} (A_n)_{j,k} f(x_k), \tag{4.6}$$

   where $(A_n)_{j,k} = \sum_{r=0}^{n} P_{n,r}(x_j) P_{n,r}(x_k)(n+1)2^{-l}$.
3. Compute the sets $\{\tilde{F}_{n,j,p,q,l} : 1 \le n \le 2^q, 0 \le j \le 2^l\}$ by

$$\begin{aligned} \tilde{F}_{n,j,p,q,l} = \Bigg(& j2^{-l} + \frac{\varphi(x_j)}{\sqrt{n}} + \frac{1}{n} - 2^{-p}\left|\tilde{D}_{n,l}(f,x_j) - f(x_j)\right|, \\ & j2^{-l} - \frac{\varphi(x_j)}{\sqrt{n}} - \frac{1}{n} + 2^{-p}\left|\tilde{D}_{n,l}(f,x_j) - f(x_j)\right|\Bigg) \cap \{k2^{-l} : 0 \le k \in \mathbb{Z}\}. \end{aligned}$$

4. Compute the sets $\{\tilde{F}_{n,p,q,l} : 1 \le n \le 2^q\}$ by

$$\tilde{F}_{n,p,q,l} = \bigcup_{j=0}^{2^l} \tilde{F}_{n,j,p,q,l}.$$

5. Compute $\tilde{F}_{p,q,l}$ by

$$\tilde{F}_{p,q,l} = \bigcup_{n=1}^{2^q} \tilde{F}_{n,p,q,l}.$$

6. Let $\tilde{S}_1(f)_{p,q,l} = \tilde{F}_{p,q,l}$ be the desired set.

Now we want to consider the convergence of this algorithm.

THEOREM 5. *Let $f \in C[0,1], S_1(f)$ and $\tilde{S}_1(f)_{p,q,l}$ be defined as above. We have*

$$\bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \left(\overline{\lim_{l\to\infty}} \tilde{S}_1(f)_{p,q,l} \right) = S_1(f) = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \underline{\lim_{l\to\infty}} \tilde{S}_1(f)_{p,q,l}. \tag{4.7}$$

Here the limit process is defined by the Hausdorff measure.

The proof of Theorem 5 can be easily obtained once we have shown the following two lemmas. We omit it here.

LEMMA 4.1. *Let $f \in C[0,1], \{\tilde{D}_{n,l}(f,x_j) : 0 \le j \le 2^l\}$ be defined by (4.6). Then we have for $0 \le j \le 2^l$, $n \in \mathbb{N}$, $l \in \mathbb{N}$:*

$$\left|\tilde{D}_{n,l}(f,x_j) - D_n(f,x_j)\right| \le \left(1 + (n+1)2^{1-l}\right) \omega_1\left(f, 2^{-l}\right) + (2n+3)2^{-l}\|f\|_\infty. \tag{4.8}$$

PROOF OF LEMMA 4.1. Let $0 \le j \le 2^l$. We have

$$\begin{aligned}
&\left|\tilde{D}_{n,l}(f,x_j) - D_n(f,x_j)\right| \\
&= \Bigg|\sum_{r=0}^{n} P_{n,r}(x_j)(n+1) \sum_{k=0}^{2^l-1} \left\{ P_{n,r}\left(\frac{k}{2^l}\right) f\left(\frac{k}{2^l}\right) 2^{-l} - \int_{k2^{-l}}^{(k+1)2^{-l}} f(t) P_{n,r}(t)\, dt \right\} \\
&\quad + \sum_{r=0}^{n} P_{n,r}(x_j) P_{n,r}(1) f(1) 2^{-l} \Bigg| \\
&\le \sum_{r=0}^{n} P_{n,r}(x_j)(n+1) \sum_{k=0}^{2^l-1} P_{n,r}\left(\frac{k}{2^l}\right) \int_{k2^{-l}}^{(k+1)2^{-l}} \left| f\left(\frac{k}{2^l}\right) - f(t) \right| dt \\
&\quad + \sum_{r=0}^{n} P_{n,r}(x_j)(n+1) \sum_{k=0}^{2^l-1} \int_{k2^{-l}}^{(k+1)2^{-l}} \left| f(t) \left(P_{n,r}\left(\frac{k}{2^l}\right) - P_{n,r}(t) \right) \right| dt + |f(1)|(1-x_j)^n 2^{-l} \\
&\le \omega_1(f, 2^{-l}) \sum_{r=0}^{n} P_{n,r}(x_j)(n+1) \sum_{k=0}^{2^l-1} P_{n,r}\left(\frac{k}{2^l}\right) 2^{-l} \\
&\quad + \sum_{r=0}^{n} P_{n,r}(x_j)(n+1) \sum_{k=0}^{2^l-1} \int_{k2^{-l}}^{(k+1)2^{-l}} \left| P_{n,r}\left(\frac{k}{2^l}\right) - P_{n,r}(t) \right| dt \|f\|_\infty + 2^{-l}\|f\|_\infty \\
&\le \omega_1\left(f, 2^{-l}\right) + \left(\omega_1\left(f, 2^{-l}\right) + \|f\|_\infty\right) \sum_{r=0}^{n} P_{n,r}(x_j)(n+1) \\
&\quad \times \sum_{k=0}^{2^l-1} \int_{k2^{-l}}^{(k+1)2^{-l}} \int_{k2^{-l}}^{(k+1)2^{-l}} |P'_{n,r}(u)|\, du\, dt + 2^{-l}\|f\|_\infty \\
&\le \omega_1\left(f, 2^{-l}\right) + \left(\omega_1\left(f, 2^{-l}\right) + \|f\|_\infty\right)(n+1)2^{1-l} + 2^{-l}\|f\|_\infty \\
&\le \left(1 + (n+1)2^{1-l}\right) \omega_1\left(f, 2^{-l}\right) + (2n+3)2^{-l}\|f\|_\infty.
\end{aligned}$$

The proof of Lemma 4.1 is complete.

LEMMA 4.2. *Let $f \in C[0,1]$. For any $p, q \in \mathbb{N}$, there exists an $L(p,q) \in \mathbb{N}$ such that for any $l \ge L(p,q)$,*

$$\tilde{F}_{p,q,l} \subset \bigcup_{n=1}^{2^q} F_{n,2^p-1} \tag{4.9}$$

and

$$\bigcup_{n=1}^{2^q} F_{n,2^p+1} \subset \tilde{F}_{p,q,l} + \left[-2^{-l}, 2^{-l}\right]. \tag{4.10}$$

The proof of Lemma 4.2 can be obtained from Lemma 4.1 and the fact that

$$\|D_n'(f)\|_\infty \le 2n\|f\|_\infty.$$

In practice, the signal f has often some global Hölder property. In this case, our algorithm can be made faster.

THEOREM 6. *Let $f \in C[0,1], \tilde{S}_1(f)_{p,q,l}$ and $S_1(f)$ be defined as above. If $\omega_1(f,t) \le M_0 t^\alpha$ for certain $1 > \alpha > 0$ and any $0 < t \le 1$. Then, for any $\delta > 0$, we can find $P, Q \in \mathbb{N}$ such that for any $p > P, q > Q, l \ge \max\{1/\alpha, 4\}q + 10 + 3/\alpha + (2/\ln 2)\ln\|f\|_\infty + 1/\alpha \ln 2 \ln M_0$,*

$$\tilde{S}_1(f)_{p,q,l} \subset \bigcup_{n=1}^{2^q} F_{n,2^p-1}$$

and

$$\bigcup_{n=1}^{2^q} F_{n,2^p+1} \subset \tilde{S}_1(f)_{p,q,l} + \left[-2^{-l}, 2^{-l}\right].$$

Theorem 6 can be obtained directly from Lemmas 4.1, 4.2. We omit the proof here.

Finally, let us mention that Theorem 1 can have the following simpler form, which can be seen from the proof of Theorem 1.

THEOREM 7. *Let $f \in C[0,1], 0 < \alpha < 1, E \subset [0,1], 2 \le r \in \mathbb{N}$. Then we have*

$$|f(x) - f(y)| \le M_f |x-y|^\alpha, \qquad x \in [0,1], \quad y \in E,$$

if and only if

$$|D_{r^n}(f,x) - f(x)| \le M_f' \left(\left(\frac{x(1-x)}{r^n} + \frac{1}{r^{2n}} \right)^{\alpha/2} + (d(x,E))^\alpha \right), \qquad x \in [0,1], \quad n \in \mathbb{N},$$

where M_f and M_f' are constants independent of x and n.

From Theorem 7, we can give another algorithm which is faster than the previous one. Since the methods are the same, we omit the details here.

Finally, we conjecture that for $f \in C[0,1]$ and $E \subset [0,1], 0 < \alpha < 1$, (1.6) holds if and only if there exists a sequence of n^{th} degree algebraic polynomials P_n ($n \in \mathbb{N}$) such that

$$|P_n(x) - f(x)| \le M \left(\left(\frac{x(1-x)}{n} \right)^\alpha + (d(x,E))^\alpha \right), \qquad x \in [0,1], \quad n \in \mathbb{N}. \tag{4.11}$$

REFERENCES

1. H. Berens and Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights, In *Approximation Theory and Functional Analysis*, (Edited by C.K. Chui), pp. 25–46, Academic Press, New York, (1990).
2. W. Chen, Z. Ditzian and K. Ivanov, Strong converse inequality for the Bernstein-Durrmeyer operators, *J. Approx. Theory* **75**, 25–43, (1993).
3. Z. Ditzian and K. Ivanov, Strong converse inequalities, *J. d'Analyse Math.* **61**, 61–111, (1993).
4. H.H. Gonska and Xin-long Zhou, A global inverse theorem on simultaneous approximation by Bernstein-Durrmeyer operators, *J. Approx. Theory* **67**, 284–302, (1991).
5. H.H. Gonska and Xin-long Zhou, The strong converse inequality for Bernstein-Kantorovich operators, submitted for publication. *Temporary reference:* Schriftenreihe des Fachbereichs Mathematik, **SM-DU-206**, Universität Duisburg 1992 (Preliminary version).

6. H.B. Knoop and Xin-long Zhou, The lower estimate for linear positive operators I, *Constr. Approx.* (to appear).
7. D.-X. Zhou, Linear operators in approximation theory and some topics in wavelets, Ph.D. Thesis, Zhejiang University, Hangzhou, (1991).
8. D.-X. Zhou, Inverse theorems for multidimensional Bernstein-Durrmeyer operators in L_p, *J. Approx. Theory* **70**, 68–93, (1992).
9. B. Lenze, On Lipschitz-type maximal functions and their smoothness spaces, *Proc. Neth. Acad. Sci., Ser. A* **91**, 53–63, (1988).
10. H. Berens and G.G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana J. Math.* **21**, 693–708, (1972).
11. Z. Ditzian and V. Totik, *Moduli of Smoothness*, (Springer Series in Computational Mathematics 9), Springer-Verlag, Berlin, (1987).
12. G.G. Lorentz, *Approximation of Functions*, Holt Rinehart and Winston, New York, (1966).
13. D.-X. Zhou, Uniform approximation by some Durrmeyer operators, *Approx. Theory Appl.* **6**, 87–100, (1990).
14. P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation Theory*, Birkhäuser, Basel, (1970).